A Note On Weinstein Conjecture*

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Abstract

In this article, we give new proofs on the some cases on Weinstein conjecture and get some new results on Weinstein conjecture.

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1 Introduction and results

Let Σ be a smooth closed oriented manifold of dimension $2n-1$. A contact form on Σ is a $1-$form such that $\lambda\wedge(d\lambda)^{n-1}$ is a volume form on Σ. Associated to λ there is the so-called Reed vectorfield X_λ defined by $i_{X_\lambda}\lambda\equiv 1,\ \ i_{X_\lambda}d\lambda\equiv 0$. The integral curve of X_λ is called *characteritcs*. There is a well-known conjecture raised by Weinstein in [17] which concerned the close Reeb orbit in a contact manifold.

Conjecture(see[17]). If (Σ,λ) is a close simply connected contact manifold with contact form λ of dimension $2n-1$, then there is a close characteristics.

Let (M,ω) be a symplectic manifold and $h(t,x)(=h_t(x))$ a compactly supported smooth function on $M\times[0,1]$. Assume that the segment $[0,1]$ is

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endowed with time coordinate t. For every function h define the $(time-dependent)$ $Hamiltonian\ vector\ field$ X_{h_t} by the equation:

$$dh_t(\eta) = \omega(\eta, X_{h_t}) \quad for\ every\ \eta \in TM \tag{1.1}$$

The flow g_h^t generated by the field X_{h_t} is called $Hamiltonian\ flow$ and its time one map g_h^1 is called $Hamiltonian\ diffeomorphism$. Now assume that H be a time independent smooth function on M and X_H its induced vector field.

Theorem 1.1 *Let (M, ω) be an exact symplectic manifold convex at infinity or with bounded geometry. Let (Σ, λ) be a contact manifold of induced type in M with induced contact form λ, i.e., there exists a vector field X transversal to Σ such that $L_X\omega = \omega$ and $\lambda = i_X\omega$, X_λ its Reeb vector field. If there exists a Hamiltonian diffeomorphism h such that $h(\Sigma) \cap \Sigma = \emptyset$, then there exists at least one close characteritics on Σ*

Corollary 1.1 *Let (M, ω) be an exact symplectic manifold which is convex at infinity or has bounded geometry(see[6]). $M \times C$ be a symplectic manifold with symplectic form $\omega \oplus \sigma$, here (C, σ) standard symplectic plane. Let $r_0 > 0$ be a fixed number and $B_{r_0}(0) \subset C$ the closed ball with radius r_0. If (Σ, λ) be a contact manifold of induced type in $M \times B_{r_0}(0)$ with induced contact form λ, i.e., there exists a vector field X transversal to Σ such that $L_X(\omega \oplus \sigma) = \omega \oplus \sigma$ and $\lambda = i_X(\omega \oplus \sigma)$, X_λ its Reeb vector field. Then there exists at least one close characteristics.*

Corollary 1.1was proved in [10] by Hofer-Viterb's method(see[8]).

Corollary 1.2 *Let M be any open manifold and $(T^*M, d\alpha)$ be its cotangent bundle. Let (Σ, λ) be a close contact manifold of induced type in T^*M. there exists at least one close characteristics on Σ.*

Corollary 1.2 generalizes the results in [9, 15, 10]. The proof of Theorem1.1 is close as in [11].

2 Lagrangian Non-squeezing

Let W be a Lagrangian submanifold in M, i.e., $\omega|W = 0$.

Definition 2.1 *Let*

$$l(M, W, \omega) = \inf\{|\int_{D^2} f^*\omega| > 0 | f : (D^2, \partial D^2) \to (M, W)\}$$

Theorem 2.1 *([12])Let (M, ω) be a closed compact symplectic manifold or a manifold convex at infinity and $M \times C$ be a symplectic manifold with symplectic form $\omega \oplus \sigma$, here (C, σ) standard symplectic plane. Let $2\pi r_0^2 < s(M, \omega)$ and $B_{r_0}(0) \subset C$ the closed disk with radius r_0. If W is a close Lagrangian manifold in $M \times B_{r_0}(0)$, then*

$$l(M, W, \omega) < 2\pi r_0^2$$

This can be considered as an Lagrangian version of Gromov's symplectic squeezing.

Corollary 2.1 *(Gromov[6])Let (V', ω') be an exact symplectic manifold with restricted contact boundary and $\omega' = d\alpha'$. Let $V' \times C$ be a symplectic manifold with symplectic form $\omega' \oplus \sigma = d\alpha = d(\alpha' \oplus \alpha_0$, here (C, σ) standard symplectic plane. If W is a close exact Lagrangian submanifold, then $l(V' \times C, W, \omega) == \infty$, i.e., there does not exist any close exact Lagrangian submanifold in $V' \times C$.*

Corollary 2.2 *Let L^n be a close Lagrangian in R^{2n} and $L(R^{2n}, L^n, \omega) = 2\pi r_0^2 > 0$, then L^n can not be embedded in $B_{r_0}(0)$ as a Lagrangian submanifold.*

3 Constructions of Lagrangian submanifolds

Let (Σ, λ) be a contact manifolds with contact form λ and X its Reeb vector field, then X integrates to a Reeb flow η_t for $t \in R^1$. Let

$$(V', \omega') = ((R \times \Sigma) \times (R \times \Sigma), d(e^a\lambda) \ominus d(e^b\lambda))$$

and

$$\mathcal{L} = \{((0, \sigma), (0, \sigma)) | (0, \sigma) \in R \times \Sigma\}.$$

Let

$$L' = \mathcal{L} \times R, L'_s = \mathcal{L} \times \{s\}.$$

Then define

$$\begin{aligned} &G' : L' \to V' \\ &G'(l') = G'(((\sigma, 0), (\sigma, 0)), s) = ((0, \sigma), (0, \eta_s(\sigma))) \end{aligned} \tag{3.1}$$

Then

$$W' = G'(L') = \{((0, \sigma), (0, \eta_s(\sigma)))|(0, \sigma) \in R \times \Sigma, s \in R\}$$

$$W'_s = G'(L'_s) = \{((0, \sigma), (0, \eta_s(\sigma)))|(0, \sigma) \in R \times \Sigma\}$$

for fixed $s \in R$.

Lemma 3.1 *There does not exist any Reeb closed orbit in (Σ, λ) if and only if $W'_s \cap W'_{s'}$ is empty for $s \neq s'$.*

Proof. First if there exists a closed Reeb orbit in (Σ, λ), i.e., there exists $\sigma_0 \in \Sigma$, $t_0 > 0$ such that $\sigma_0 = \eta_{t_0}(\sigma_0)$, then $((0, \sigma_0), (0, \sigma_0)) \in W'_0 \cap W'_{t_0}$. Second if there exists $s_0 \neq s'_0$ such that $W'_{s_0} \cap W'_{s'_0} \neq \emptyset$, i.e., there exists σ_0 such that

$$((0, \sigma_0), (0, \eta_{s_0}(\sigma_0)) = ((0, \sigma_0), (0, \eta_{s'_0}(\sigma_0)),$$

then $\eta_{(s_0 - s'_0)}(\sigma_0) = \sigma_0$, i.e., $\eta_t(\sigma_0)$ is a closed Reeb orbit.

Lemma 3.2 *If there does not exist any closed Reeb orbit in (Σ, λ) then there exists a smooth Lagrangian injective immersion $G' : W' \to V'$ with $G'(((0, \sigma), (0, \sigma)), s) = ((0, \sigma), (0, \eta_s(\sigma)))$ such that*

$$G'_{s_1, s_2} : \mathcal{L} \times (-s_1, s_2) \to V' \tag{3.2}$$

is a regular exact Lagrangian embedding for any finite real number s_1, s_2, here we denote by $W'(s_1, s_2) = G'_{s_1,s_2}(\mathcal{L} \times (s_1, s_2))$.

Proof. One check

$$G'^*((e^a\lambda - e^b\lambda)) = \lambda - \eta(\cdot, \cdot)^*\lambda = \lambda - (\eta_s^*\lambda + i_X\lambda ds) = -ds \tag{3.3}$$

since $\eta_s^*\lambda = \lambda$. This implies that G' is an exact Lagrangian embedding, this proves Lemma 3.2.

Now we modify the above construction as follows:

$$
\begin{aligned}
&F' : \mathcal{L} \times R \times R \to (R \times \Sigma) \times (R \times \Sigma) \\
&F'(((0,\sigma),(0,\sigma)), s, b) = ((0,\sigma),(b,\eta_s(\sigma)))
\end{aligned}
\tag{3.4}
$$

Now we embed a elliptic curve E long along $s-axis$ and thin along $b-axis$ such that $E \subset [-s_1, s_2] \times [0, \varepsilon]$. We parametrize the E by t.

Lemma 3.3 *If there does not exist any closed Reeb orbit in* (Σ, λ)*, then*

$$
\begin{aligned}
&F : \mathcal{L} \times S^1 \to (R \times \Sigma) \times (R \times \Sigma) \\
&F(((0,\sigma),(0,\sigma)), t) = ((0,\sigma),(b(t),\eta_{s(t)}(\sigma)))
\end{aligned}
\tag{3.5}
$$

is a compact Lagrangian submanifold. Moreover

$$
l(V', F(\mathcal{L} \times S^1, d(e^a\lambda - e^b\lambda)) = area(E) \tag{3.6}
$$

Proof. We check that

$$
F^*(e^a\lambda \ominus e^b\lambda) \;\; = \;\; -e^{b(t)}ds(t) \tag{3.7}
$$

So, F is a Lagrangian embedding.

If the circle C homotopic to $C_1 \subset \mathcal{L} \times s_0$ then we compute

$$
\int_C F^*(e^a\lambda - e^b\lambda) = \int_{C_1} F^*(e^0\lambda - e^0\lambda) = 0. \tag{3.8}
$$

since $\lambda - \lambda | C_1 = 0$ due to $C_1 \subset \mathcal{L}$. If the circle C homotopic to $C_1 \subset l_0 \times S^1$ then we compute

$$
\int_C F^*(e^a\lambda - e^b\lambda) = \int_{C_1} (-)e^b ds = n(area(E)). \tag{3.9}
$$

This proves the Lemma.

Gromov's figure eight construction: First we note that the construction of section 3.1 holds for any symplectic manifold. Now let (M,ω) be an exact symplectic manifold with $\omega = d\alpha$. Let $\Sigma = H^{-1}(0)$ be a regular and close smooth hypersurface in M. H is a time-independent Hamilton function.

Set $(V', \omega') = (M \times M, \omega \ominus \omega)$. If there does not exist any close orbit for X_H in (Σ, X_H), one can construct the Lagrangian submanifold L as in section 3.1, let $W' = L$. Let $h_t = h(t, \cdot) : M \to M$, $0 \le t \le 1$ be a Hamiltonian isotopy of M induced by hamilton fuction H_t such that $h_1(\Sigma) \cap \Sigma = \emptyset$, $|H_t| \le C_0$. Let $\bar{h}_t = (id, h_t)$. Then $F'_t = \bar{h}_t : W' \to V'$ be an isotopy of Lagrangian embeddings. As in [6], we can use symplectic figure eight trick invented by Gromov to construct a Lagrangian submanifold W in $V = V' \times R^2$ through the Lagrange isotopy F' in V', i.e., we have

Proposition 3.1 *Let V', W' and F' as above. Then there exists a weakly exact Lagrangian embedding $F : W' \times S^1 \to V' \times R^2$ with $W = F(W' \times S^1)$ is contained in $M \times M \times B_R(0)$, here $4\pi R^2 = 8C_0$ and*

$$l(V', W, \omega) = area(M'_0) = A(T). \tag{3.10}$$

Proof. Similar to [6, $2.3B'_3$].

Example. Let M be an open manifold and $(T^*M, p_i dq_i)$ be the cotangent bundle of open manifold with the Liouville form $p_i dq_i$. Since M is open, there exists a function $g : M \to R$ without critical point. The translation by $tTdg$ along the fibre gives a hamilton isotopy of T^*M : $h_t^T(q,p) = (q, p + tTdg(q))$, so for any given compact set $K \subset T^*M$, there exists $T = T_K$ such that $h_1^T(K) \cap K = \emptyset$.

3.1 Proof on Theorem 1.1

Since (Σ, λ) be a close contact manifold of induced type in M with induced contact form λ, then by the well known theorem that the neighbourhood $(U(\Sigma), \omega)$ of Σ is symplectomorphic to $([-\varepsilon, \varepsilon] \times \Sigma, de^a\lambda)$ for small ε. So, by Proposition 3.1, we have a close Lagrangian submanifold $F(\mathcal{L} \times S^1)$ contained in $M \times M \times B_R(0)$. By Lagrangian squeezing theorem, i.e., Theorem 2.1, we have

$$l((M \times M \times C), F(\mathcal{L} \times S^1, \omega \oplus \omega) = area(E) \le 2\pi R^2. \tag{3.11}$$

If $s_2 - s_1$ large enough, $area(E) > 2\pi R^2$. This is a contradiction. This contradiction shows there exists at least one close characteristics.

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